UDC 539.3

ON THE SOLUTION OF AN INTEGRAL EQUATION OF THE CONTACT PROBLEM IN THE THEORY OF ELASTICITY FOR ROUGH BODIES

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A method is proposed for solving the integral equation for the contact of rough bodies by using a special approximation of the required solution.

1. The contact between the surfaces of machine components is discrete in nature because of microgeometric surface roughnesses, and the surfaces in contact are called rough. Following [1], we consider the elastic displacements for rough bodies to equal the sum of the displacements originating because of the strain of the whole elastic body and certain additional displacements governed by the surface structure of the body.

On the basis of experimental data (*) it is assumed that the additional normal displacement is proportional to the normal pressure to some degree. Hence, the plane and three-dimensional contact problems for rough bodies should be described by the equations

$$Kp^\alpha(x) + \theta \int_{-a}^{a} \ln \frac{1}{|t-x|} p(t)\,dt = f(x) \tag{1.1}$$

$$Kp^\alpha(x, y) + \theta \iint_\Omega \frac{p(t, \tau)}{\sqrt{(x-t)^2 + (y-\tau)^2}}\,dt\,d\tau = f(x, y) \tag{1.2}$$

$$\theta = \frac{1-\mu_1^2}{\pi E_1} + \frac{1-\mu_2^2}{\pi E_2}$$

Here μ_i, E_i are the elastic constants of the bodies in contact, a and Ω govern the nominal contact sections, $f(x)$ and $f(x, y)$ are functions determined by the shape, angle of rotation and displacement of the bodies in contact, and K, α are constants dependent on the elastic characteristics of the material and the surface properties of the bodies in contact, where $0.3 \leqslant \alpha \leqslant 1$. The existence of bounded solutions of (1.1) and (1.2) in a closed bounded domain follows from theorems [2] on the solvability of a nonlinear equation of the form

$$y(x) = \int_B k(x, t)\,\Phi(t, y(t))\,dt$$

to which (1.1) and (1.2) reduce by the displacement $y = f - Kp^\alpha$. The bounded solution exists for the one-dimensional integral equations if:

a) For any fixed $x \in [-a, a]$, $x^* \in [-a, a]$ $k(x, t)$ is summable in t and

$$\lim_{x \to x^*} \int_{-a}^{a} |k(x, t) - k(x^*, t)|^2\,dt = 0$$

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*) Rabinovich, A. S., Contact Problems of Elasticity Theory for Rough Bodies. Summary of a Kandidat Dissertation, Moscow State Univ., 1975.

b) $\Phi(x, y)$ satisfies the condition $|\Phi(t, z)| \leqslant \mu^{-1}|z| + c$, where $c > 0$ and $\mu > \lambda_0$; λ_0 is the greatest positive eigenvalue for $|k(x, t)|$; if there is none, then $\lambda_0 = 0$. Conditions (a) and (b) are satisfied for (1.1) for

$$c = \Theta M^{1/\alpha} K^{-1/\alpha}, \quad \mu = K^{1/\alpha}\alpha [\Theta M^{1/\alpha - 1} Q]^{-1}$$
$$Q = [(1 + q)^{1/\alpha} - 1] q^{-1}, \quad M = \max_{x \in [a, a]} |f(x)|$$

if

$$|y| \leqslant qM, \quad \mu > 2a \ln(e/a), \; q < 1, \; 2a \leqslant 1$$

(λ_0 is determined from the equation $\ln a \sin(\pi a/\lambda) - \mathrm{si}(\pi a/\lambda) = 0$ and is within the range $|\lambda_0| \leqslant 2a|\ln a - 1|$).

If $p(a) = 0$, compliance with the condition $\mu > 2a$ is necessary. Analogous reasoning is performed for Eq. (1.2). In this case the eigenfunctions and eigenvalues for the kernel depend substantially on the shape of the domain.

2. Let us represent the solution (1.1) and (1.2) in the form: (*)

$$p_N(x) = p_0(x) S_N(x), \quad S_N(x) = \sum_{k=1}^{N} c_k \omega_k (x - x_k) \tag{2.1}$$

$$p_N(x, y) = p_0(x, y) \sum_{k=1}^{N} a_k d_k \omega_{1k}(x - x_k)\, \omega_{2k}(y - y_k) \tag{2.2}$$

where a_k and c_k are unknown coefficients determined from the conditions that (2.1) and (2.2) satisfy (1.1), (1.2) at the sites x_k (x_k, y_k), and ω_{ik} are functions of the form

$$\frac{1}{b_{ik}} \exp\left[-\left(\frac{t_i - t_{ik}}{b_{ik}}\right)^2\right], \quad \theta(h_{ik} - |t_i - t_{ik}|)\left(1 - \frac{|t_i - t_{ik}|}{q_{ik}}\right)$$

which are δ-sequences of functions [3], and $\theta(x)$ is the unit function

$$h_{ik} = t_{ik+1} - t_{ik}, \quad q_{ik} = \begin{cases} b_{ik+1}, & t_i > t_{ik} \\ b_{ik}, & t_i \leqslant t_{ik} \end{cases}$$

The notation $t_1 = x$, $t_2 = y$, $t_{1k} = x_k$, $t_{2k} = y_k$ is introduced; x_k (x_k, y_k) are points of discretization of the domain which depend on its shape; d_k are known constants playing the part of the weight in the cubic formula; p_0 is a function taking into account the behavior of the density p on the edges of the stamp. If the computations are executed without isolating singularities, it is assumed that $p_0 = 1$.

The functions $p(x)$, $p(x, y)$, which are continuous in a closed bounded domain Ω^*, can approach functions of the form (2.1) and (2.2) for sufficiently large N and sufficiently small b_{ik} for any inner point of Ω^*. The minimum estimate of the deviation is hence obtained for $b_{ik} = h_{ik}$.

This representation permits obtaining an explicit expression for the pressure under the base of a rough stamp of arbitrary shape. Solutions of problems by the method mentioned for $N \leqslant 10$ are practically in agreement with those published in [1, 3].

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*) Romanchik, V. S., On a method of approximating functions and its application to the solution of integral equations of the first kind. Republic Conference of Belorussian Mathematicians. Abstract of Reports. Belorussia University Press, Minsk, 1975.

3. For definiteness, let us examine the integral equation describing the problem of the pressure of a parabolic stamp on an elastic half-plane [3]

$$Kp^{\alpha}(y) + \int_{-a}^{a} p(t)\ln\left|\frac{t-a}{t-y}\right| dt = c(a^2 - y^2) \tag{3.1}$$

We append an equation expressing the equilibrium condition of the stamp

$$\int_{-a}^{a} p(t)\,dt = P \tag{3.2}$$

to this equation. We represent the solution of (3.1) in the form

$$p(y) = a\sqrt{1-x^2}\,S_N(x) \tag{3.3}$$
$$x = \frac{y}{a}, \quad \omega_k = \frac{1}{b_k}\exp\left[-\left(\frac{x-x_k}{b_k}\right)^2\right]$$

Because of symmetry of the stamp we assume

$$x_k = (k - {}^1/_2)H, \quad H = 1/N, \quad k = 1, N; \quad 0 \leqslant x \leqslant 1$$

From (3.2) we obtain

$$a = \left\{P\left[\sum_{k=1}^{N} c_k \int_{-1}^{1} \sqrt{1-t^2}\,\omega_k(t-x_k)\,dt\right]^{-1}\right\}^{1/2} \tag{3.4}$$

where P is a compressive force. After substituting (3.3) and (3.4) into (3.1) and a collocation at the sites x_k we obtain a system of algebraic equations to determine the unknown constants a and c_k

$$f_i = 0, \quad 1 \leqslant i \leqslant N \tag{3.5}$$
$$f_i = K\left\{P\left[\sum_{k=1}^{N} c_k \int_{-1}^{1}\sqrt{1-t^2}\,\omega_k(t-x_k)dt\right]^{-1}\right\}^{\alpha/2-1}(1-x_i^2)^{\alpha/2}S_N^{\alpha}(x_i) +$$
$$\sum_{k=1}^{N} c_k \int_{-1}^{1} \omega_k(t-x_k)\sqrt{1-t^3}\ln\left|\frac{1-t}{t-x_i}\right| dt - c(1-x_i^2)$$

The system obtained is solved by iteration and the successive approximations are determined by the formulas (a is defined by means of (3.4))

$$c_i^{n+1} = c_i^n - \sum_{j=1}^{N} [r_{ij}^n]^{-1} f_j^n, \quad i = 1, N \tag{3.6}$$
$$r_{ij}^n = r_{ij}\,|\,c_j = c_j^n, \quad f_i^n = f_i\big|_{c_i = c_i^n}, \quad i = 1, N; \quad j = 1, N$$

$$r_{ij} \equiv \frac{\partial f_i}{\partial c_j} = Ka^{\alpha/2}(1-x_i^2)^{\alpha/2}S_N^{\alpha}(x_i)\left[\frac{\alpha\omega_j(x_i - x_j)}{aS_N(x_i)} + \right. \tag{3.7}$$
$$\left.\frac{(1-\alpha/2)}{P}\int_{-1}^{1}\sqrt{1-t^2}\,\omega_j(t-t_j)\,dt\right] +$$
$$\int_{-1}^{1} \omega_j(t-t_j)\sqrt{1-t^2}\ln\left|\frac{t-1}{t-x_i}\right| dt$$

Let us convert the improper integrals in (3.5) and (3.7) to the following form:

$$I = \int_{-1}^{1} \omega_j (t - t_j) \sqrt{1 - t^2} \ln \left| \frac{t-1}{t - x_i} \right| dt =$$

$$\int_{0}^{1} \omega_j (t - t_j) \sqrt{1 - t^2} \left[\ln \left| \frac{t-1}{t - x_i} \right| + \ln \frac{t+1}{t + x_i} \right] dt$$

To evaluate integrals of the form

$$\int_0^1 \sqrt{1 - t^2}\, \varphi(t) \ln (t - x)\, dt$$

the range of integration is separated into segments $[0, x]$, $[x, 1]$, in which the substitution $t = xu$, $t = 1 - (1 - x)\, u$ has been performed, respectively, and then quadrature formulas have been used to evaluate the integrals [4]

$$\int_0^1 x^{\alpha} \ln \frac{e}{x} f(x)\, dx, \quad \int_0^1 f(x) \ln \frac{1}{x}\, dx, \quad \int_0^1 \sqrt{1 - t}\, f(t)\, dt$$

The results of calculations on the computer are presented in Fig. 1 and in Table 1. Curves 1—3 correspond to the values $K = 0$, $K = 0.18$, $K = 0.36$ for $\alpha = 0{,}6666$, $c = 15.7$ and $P = 0{,}157$. The solutions differ by 1—2% from those presented in [3]. The values obtained for the half-width of the contact section a equal 0.1, 0.149, 0.176. Values of $p\,(x_i)$ presented in [3] and obtained by the proposed method are presented in columns A and B of Table 1 for $c = 0.863\cdot 10^5$ kgf/cm^3, $P = 0.86\cdot 10^5$ kgf/cm, $K = 0$, $P = 0.8\cdot 10^5$ kgf/cm, $K = 0.1726$, $P = 0.7\cdot 10^5$kgf/cm, $K = 1{,}726$, $P = 0.7\cdot 10^5$ kgf/cm, $K = 17.26$. The solutions differ by not more than 2% from those known. The values $a = 1.0$, $a = 1.04$, $a = 1.30$, $a = 2.34$ are obtained corresponding to 1—4.

Table 1

x/a	1		2		3		4	
	A	B	A	B	A	B	A	B
0.0	0.550	0.550	0.485	0.505	0.385	0.385	0.225	0.227
0.2	0.539	0.538	0.483	0.497	0.373	0.370	0.216	0.219
0.4	0.504	0.500	0.446	0.459	0.334	0.332	0.191	0.192
0.6	0.440	0.433	0.378	0.390	0.266	0.265	0.147	0.148
0.8	0.330	0.320	0.264	0.269	0.164	0.161	0.086	0.085

4. Let us consider the problem of the pressure of a flat stamp on an elastic rough half-plane. In this case (1.1) and the equilibrium equation are

$$K p^{\alpha}(y) + \int_{-a}^{a} p(t) \ln \left| \frac{1}{t - y} \right| dt = c \tag{4.1}$$

$$\int_{-a}^{a} p(t)\, dt = P \tag{4.2}$$

We make the substitution $p(y) = p_1(ax)\,c$ and represent the solution of the problem in the form

$$p_1(ax) = S_N(x) \tag{4.3}$$

We determine the constant c from (4.2)

$$c = P\left[a \sum_{k=1}^{N} a_k \int_{-1}^{1} \omega_k(t - t_k)\,dt\right]^{-1} \tag{4.4}$$

We select the function

$$\omega_k = \theta(h_k - |x - x_k|)\left(1 - \frac{|x - x_k|}{b_k}\right)$$

as ω_k.

Such a representation possesses the advantage that the integrals in (4.1) and (4.4) are evaluated exactly. After substituting (4.3) and (4.4) into (4.1) and a collocation at the sites, we obtain a system of nonlinear equations which we solve analogously.

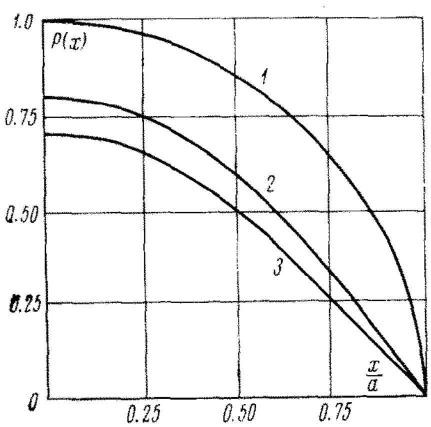

Fig. 1

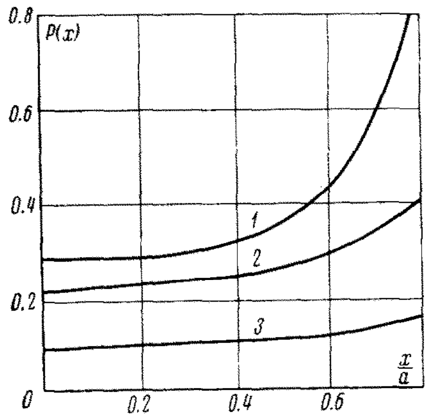

Fig. 2

The results of the calculations are presented in Fig. 2. Curves 1—3 correspond to the values $P = 0.6$, $K = 1{,}364$, $\alpha = 1$, $a = 1$; $P = 0{,}6$, $K = 0.361$, $\alpha = 0.5$, $a = 0.7$; $P = 0.173$, $K = 1.364$. $\alpha = 0.8$, $a = 0.7$. The following values of c are obtained, respectively, from 1—3: $c = 0.932$; $c = 0.934$; $c = 0.269$. For $K = 3{,}1415$, $\alpha = 1$ the results of the calculations differ insignificantly from those presented in [1].

5. As a criterion for cutting off the iteration process we can select the condition of closeness of two adjacent approximations. The difference between adjacent approximations did not exceed 10^{-6} for calculations on the "Minsk-32" electronic computer. The value of the residual is hence $r_n \leqslant 10^{-9}$.

According to [5], the Newtonian process of solving the system of algebraic equations $F(c) = 0$, given in the sphere $S(c_0, R)$, reduces to the solution c^* given in a sphere $s(c_0, r_0)$ and the following estimate is true

$$\| c^n - c^* \| \leqslant \frac{1}{2^n}(2h_0)^{2^n - 1}\eta$$

if

$$h_0 \leqslant 1/2, \quad R \geqslant r_0 = (1 - \sqrt{1 - 2h_0})\,\eta_0 h_0^{-1} \tag{5.1}$$

Here $h_0 = b_0 L \eta_0$, b_0 is the estimate of the norm of the inverse matrix $\| F'^{-1} \| \leqslant b_0$, η_0 characterizes the selection of the initial approximation $| F'^{-1}(c_0) F(c_0) | \leqslant \eta_0$, and L is the Lipschitz constant for $F'(c)$. Selected as the norm of the matrix is

$$\| F \| = \max_{i \leqslant N} \sum_{j=1}^{N} | a_{ij} |$$

and the quantities L, b_0, η_0 are evaluated by using an electronic computer. Thus, for instance, the values $L = 100$, $b_0 = 0.5$ are obtained for (3.1) with $K = 1.726$. The initial approximation is selected so that $\eta_0 \leqslant 0.01$ and that the conditions (5.1) are satisfied and the inequality holds

$$\max_{k \leqslant N} | c_k^n - c_k | \leqslant \varepsilon \tag{5.2}$$

No instability in the iteration process was noted in performing the computations; the iteration process converged for arbitrary initial approximations, which holds for concave operators.

By using the inequality (5.2) an estimate can be deduced on the closeness of

$$p_N^{(n)}(x) = \sum_{k=1}^{N} c_k^n \omega_k (x - x_k) \quad \text{to} \quad p_N = S_N(x)$$

to the exact solution of the equation

$$K p_N^{\alpha}(x_i) - \int_{-a}^{a} k(x_i, t)\, p_N(t)\, dt = f(x_i)$$

Multiplying (5.2) by $\omega_k(x - x_k)$, summing over k and going over to the max with respect to x, we obtain

$$\max_x \Big| \sum_{k=1}^{N} c_k^n \omega_k (x - x_k) - S_N(x) \Big| \leqslant \varepsilon \max_x \sum_{k=1}^{N} \omega_k (x - x_k)$$

or

$$| p_N^{(n)}(x) - p_N(x) | \leqslant \varepsilon_1$$

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